

On the Reducibility of Scalar Generalized Verma Modules of Abelian Type

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Abstract

A parabolic subalgebra \mathfrak{p} of a complex semisimple Lie algebra \mathfrak{g} is called a parabolic subalgebra of abelian type if its nilpotent radical is abelian. In this paper, we provide a complete characterization of the parameters for scalar generalized Verma modules attached to parabolic subalgebras of abelian type such that the modules are reducible. The proofs use Jantzen's simplicity criterion, as well as the Enright-Howe-Wallach classification of unitary highest weight modules.

1 Introduction

Let \mathfrak{g} be a complex semisimple Lie algebra, and fix a Cartan subalgebra \mathfrak{h} . Denote by Φ (respectively, Φ^+) the set of roots (respectively, positive roots) of $(\mathfrak{g}, \mathfrak{h})$. Let \mathfrak{p} be a maximal parabolic subalgebra of \mathfrak{g} with standard Levi decomposition $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ with respect to (\mathfrak{h}, Φ^+) , where \mathfrak{l} is a Levi factor and \mathfrak{u}_+ is the nilpotent radical. Let $\Phi_{\mathfrak{l}}$ be the set of roots of $(\mathfrak{l}, \mathfrak{h})$, and put $\Phi_{\mathfrak{l}}^+ = \Phi_{\mathfrak{l}} \cap \Phi^+$. If $\lambda \in \mathfrak{h}^*$ is $\Phi_{\mathfrak{l}}^+$ -dominant integral, let F_{λ} be the finite-dimensional complex simple \mathfrak{l} -module with highest weight λ . By letting the nilpotent radical \mathfrak{u}_+ act by 0, F_{λ} is also a module of the parabolic subalgebra \mathfrak{p} . Now the generalized Verma module of \mathfrak{g} attached to \mathfrak{p} with the parameter λ is defined to be

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_{\lambda}$$

where $U(\mathfrak{g})$ (respectively, $U(\mathfrak{p})$) is the universal enveloping algebra of \mathfrak{g} (respectively, \mathfrak{p}). When $\dim_{\mathbb{C}} F_{\lambda} = 1$, $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is called a scalar generalized Verma module; when \mathfrak{p} is a Borel subalgebra, it is just a Verma module. As is known, generalized Verma modules form a fundamental and distinguished class of objects in the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$. A universal property is that each highest weight module in $\mathcal{O}^{\mathfrak{p}}$ can be covered by a generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ for some λ . More details about generalized Verma modules can be found in [H].

The reducibility of generalized Verma modules is an interesting problem, which is much more complicated than the problem for Verma modules. The study of this problem has a long history. In 1977, a simplicity criterion for generalized Verma modules was shown by J. C. Jantzen in [J], and it remains one of the most well-known and widely used results along these lines. After that, T. Enright, R. Howe, and N. Wallach worked out the parameters of reducible generalized Verma modules related to unitary highest weight modules in 1983 [EHW]. Recently, A. Kamita

used b -functions to describe the reducibility of generalized Verma modules in [Ka]. Apart from directly studying the reducibility problem, many mathematicians studied homomorphisms between generalized Verma modules, which gave some results on reducibility of generalized Verma modules indirectly, e.g., [B], [F], [G], and [M].

A parabolic subalgebra \mathfrak{p} of a complex semisimple Lie algebra \mathfrak{g} is called a parabolic subalgebra of abelian type if its nilpotent radical is abelian. We now explain how to characterize the parabolic subalgebras of abelian type in simple Lie algebras in terms of Hermitian symmetric pairs. Suppose that G is a connected real simple Lie group with center Z , and let K be a closed maximal subgroup of G with K/Z compact. Let \mathfrak{g} be the complexified Lie algebra of G . A unitary representation (π, V) of G such that the underlying (\mathfrak{g}, K) -module is a simple quotient of a Verma module of \mathfrak{g} is called a unitary highest weight module. Harish-Chandra showed that G admits non-trivial unitary highest weight modules precisely when (G, K) is a Hermitian symmetric pair ([HC1] and [HC2]). Now denote by \mathfrak{l} the complexified Lie algebra of K . Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{l} ; then our assumptions on G imply that \mathfrak{h} is also a Cartan subalgebra of \mathfrak{g} . Since $(\mathfrak{g}, \mathfrak{l})$ is a Hermitian symmetric pair, we may choose a simple root system Δ of $(\mathfrak{g}, \mathfrak{h})$ such that the standard Borel subalgebra \mathfrak{b} satisfying that $\mathfrak{p} := \mathfrak{l} + \mathfrak{b}$ is a parabolic subalgebra of \mathfrak{g} . According to the classification of the Hermitian symmetric pairs [W], the nilpotent radical of \mathfrak{p} must be abelian and hence \mathfrak{p} is a parabolic subalgebra of abelian type. Conversely, suppose that \mathfrak{g} is a complex simple Lie algebra and \mathfrak{p} is a parabolic subalgebra of abelian type of \mathfrak{g} , and then \mathfrak{p} is automatically a maximal parabolic subalgebra. It follows from [RRS] that there exists a real form $\mathfrak{g}_{\mathbb{R}}$ of \mathfrak{g} such that $G_{\mathbb{R}}/(G_{\mathbb{R}} \cap P)$ is a Hermitian symmetric space, where $G_{\mathbb{R}}$ and P are the subgroups of the adjoint group $G = \text{Int}(\mathfrak{g})$ with Lie algebras $\mathfrak{g}_{\mathbb{R}}$ and \mathfrak{p} . The group $K := G_{\mathbb{R}} \cap P$ is a maximal compact subgroup of $G_{\mathbb{R}}$, and its complexified Lie algebra gives the Levi factor, denoted by \mathfrak{l} , of \mathfrak{p} .

If \mathfrak{p} is a parabolic subalgebra of abelian type, we call a generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ attached to \mathfrak{p} a generalized Verma module of abelian type. In [EHW], T. Enright, R. Howe, and N. Wallach worked out all the parameters λ such that the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ of abelian type is reducible and its unique simple quotient $L(\lambda)$ is unitarizable. However, if $L(\lambda)$ is not unitarizable, it is not known for which λ $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple. We shall answer this question when $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is a scalar generalized Verma module, i.e., $F_{\lambda} = \mathbb{C}_{\lambda}$ is a one-dimensional complex \mathfrak{l} -module, which is equivalent to saying that $(\lambda, \alpha) = 0$ for all $\alpha \in \Phi_{\mathfrak{l}}$, where $(-, -)$ is the inner product on \mathfrak{h}^* induced by the Killing form of \mathfrak{g} . We shall recall the techniques in [EHW] in Section 2.2.

Let γ denote the unique maximal root in Φ^+ . Denote by ζ the unique element in \mathfrak{h}^* such that $(\zeta, \alpha) = 0$ for all $\alpha \in \Phi_{\mathfrak{l}}$ and $\frac{2(\zeta, \gamma)}{(\gamma, \gamma)} = 1$. It is obvious that the parameters λ of scalar generalized Verma modules must satisfy $\lambda = c\zeta$ for some $c \in \mathbb{C}$.

Write $\Phi_{\mathfrak{u}} := \Phi \setminus \Phi_{\mathfrak{l}}$, and denote by Δ the simple root system of Φ^+ . Then each standard maximal parabolic subalgebra with respect to (\mathfrak{h}, Δ) is determined by the unique simple root in $\Delta_{\mathfrak{u}} := \Phi_{\mathfrak{u}} \cap \Delta$. Now we can state our main result.

Theorem 1.1. *Let \mathfrak{g} be a complex simple Lie algebra, and let \mathfrak{p} be a parabolic subalgebra of abelian type of \mathfrak{g} . Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{p}$, and choose a simple root system Δ for $(\mathfrak{g}, \mathfrak{h})$ such that \mathfrak{p} contains the standard Borel subalgebra with respect to (\mathfrak{h}, Δ) . Then the parameters $\lambda \in \mathfrak{h}^*$ such that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is a reducible scalar generalized Verma module are precisely given case*

by case, according to the Hermitian symmetric pairs of compact type, as follows.

- (i) $(SU(p+q), S(U(p) \times U(q)))$ for $p, q \geq 1$: $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq p+q-1\}$; $\Delta_u = \{e_p - e_{p+1}\}$; $\zeta = \frac{q}{p+q} \sum_{i=1}^p e_i - \frac{p}{p+q} \sum_{j=p+1}^{p+q} e_j$; $\lambda = c\zeta$ with $c \in 1 - \min\{p, q\} + \mathbb{Z}_{\geq 0}$.
- (ii) $(Sp(n), U(n))$ for $n \geq 2$: $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2e_n\}$; $\Delta_u = \{2e_n\}$; $\zeta = \sum_{i=1}^n e_i$; $\lambda = c\zeta$ with $c \in \frac{1-n}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}$.
- (iii) $(SO(2n+1), SO(2) \times SO(2n-1))$ for $n \geq 2$: $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_n\}$; $\Delta_u = \{e_1 - e_2\}$; $\zeta = e_1$; $\lambda = c\zeta$ with $c \in \mathbb{Z}_{\geq 0} \cup (\frac{3}{2} - n + \mathbb{Z}_{\geq 0})$.
- (iv) $(SO(2n), SO(2) \times SO(2n-2))$ for $n \geq 2$: $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$; $\Delta_u = \{e_1 - e_2\}$; $\zeta = e_1$; $\lambda = c\zeta$ with $c \in 2 - n + \mathbb{Z}_{\geq 0}$.
- (v) $(SO(2n), U(n))$ for $n \geq 2$: $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$; $\Delta_u = \{e_{n-1} + e_n\}$; $\zeta = \frac{1}{2} \sum_{i=1}^n e_i$; $\lambda = c\zeta$ with $c \in 2[\frac{3-n}{2}] + \mathbb{Z}_{\geq 0}$ where $[x]$ denotes the largest integer not greater than $x \in \mathbb{R}$.
- (vi) $(E_{6(-78)}, SO(2) \times SO(10))$: $\Delta = \{\alpha_i \mid 1 \leq i \leq 6\}$ with $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_i = e_{i-1} - e_{i-2}$ ($3 \leq i \leq 6$); $\Delta_u = \{\alpha_1\}$; $\zeta = \frac{2}{3}(-e_6 - e_7 + e_8)$; $\lambda = c\zeta$ with $c \in -3 + \mathbb{Z}_{\geq 0}$.
- (vii) $(E_{7(-78)}, SO(2) \times E_{6(-78)})$: $\Delta = \{\alpha_i \mid 1 \leq i \leq 7\}$ with α_i ($1 \leq i \leq 6$) same as in (vi) and $\alpha_7 = e_6 - e_5$; $\Delta_u = \{\alpha_7\}$; $\zeta = e_6 - \frac{1}{2}e_7 + \frac{1}{2}e_8$; $\lambda = c\zeta$ with $c \in -8 + \mathbb{Z}_{\geq 0}$.

Remark 1.2. An observation from Theorem 1.1 is that in each case, the set of parameters for which the corresponding scalar generalized Verma modules of abelian type are reducible constitutes a finite set and a semi-infinite arithmetic progression.

Remark 1.3. Although Theorem 1.1 only discusses complex simple Lie algebras, people may deduce the results for complex semisimple Lie algebras immediately. In fact, suppose that $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ denotes the decomposition of a finite-dimensional complex semisimple Lie algebra into its simple ideals, and \mathfrak{p} is a parabolic subalgebra of \mathfrak{g} . Then

- $\mathfrak{p} = \bigoplus_j \mathfrak{p}_j$, with each \mathfrak{p}_j a parabolic subalgebra of \mathfrak{g}_j .
- \mathfrak{p} has abelian (respectively, nilpotent) radical if and only if each \mathfrak{p}_j has abelian (respectively, nilpotent) radical.
- $\lambda = \sum_j \lambda_j$ is a dominant integral highest weight with respect to \mathfrak{p} if and only if each λ_j is dominant integral highest weight with respect to \mathfrak{p}_j .
- For the above data, $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) = \bigotimes_j M_{\mathfrak{p}_j}^{\mathfrak{g}_j}(\lambda_j)$.
- $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is a scalar (simple) generalized Verma module if and only if each $M_{\mathfrak{p}_j}^{\mathfrak{g}_j}(\lambda_j)$ is a scalar (simple) generalized Verma module.

From these points, we actually obtain the parameters of all the reducible scalar generalized Verma modules of abelian type for complex semisimple Lie algebras.

The reducibility problem can be studied for scalar generalized Verma modules attached to arbitrary parabolic algebras. Concretely, let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of a semisimple Lie algebra \mathfrak{g} , where \mathfrak{l} is a Levi factor and \mathfrak{u}_+ is the nilpotent radical. Put $\mathfrak{u}_0 = \mathfrak{u}_+$ and $\mathfrak{u}_k = [\mathfrak{u}_+, \mathfrak{u}_{k-1}]$ for positive integer k . We call \mathfrak{u}_k the k th-step of \mathfrak{u}_+ for a nonnegative integer k . The nilpotent Lie algebra \mathfrak{u}_+ is called k -step nilpotent if $\mathfrak{u}_{k-1} \neq 0$ and $\mathfrak{u}_k = 0$. If the nilpotent radical \mathfrak{u}_+ of the parabolic subalgebra \mathfrak{p} is k -step nilpotent, then we say that \mathfrak{p} is a parabolic subalgebra of k -step nilpotent type. In particular, if $\mathfrak{u}_0 = \mathfrak{u}_+ = 0$, then $\mathfrak{p} = \mathfrak{g}$, and hence generalized Verma modules attached to \mathfrak{p} are just finite-dimensional complex simple modules of \mathfrak{g} , whose reducibility problems are well-known [Kn, Theorem 5.5]. If \mathfrak{p} is a parabolic subalgebra of 1-step nilpotent type, then it is just a parabolic subalgebra of abelian type, the scalar generalized Verma modules attached to which are what this paper handles. As for a parabolic subalgebra \mathfrak{p} of 2-step nilpotent type, if $\dim_{\mathbb{C}} \mathfrak{u}_1 = 1$, then \mathfrak{p} is called a parabolic subalgebra of 2-step nilpotent Heisenberg type; else, it is called a parabolic subalgebra of 2-step nilpotent non-Heisenberg type. T. Kubo investigated and solved the reducibility problem for scalar generalized Verma modules associated to exceptional simple Lie algebras and maximal parabolic subalgebras of 2-step nilpotent non-Heisenberg type in [Ku]. Moreover, according to T. Kubo, generalized Verma modules associated to exceptional simple Lie algebras and maximal parabolic subalgebras of 2-step nilpotent Heisenberg type were studied by R. Zierau, but the results were not published.

The paper is organized as follows. In Section 2, we recall Jantzen's criterion for simplicity of generalized Verma modules attached to maximal parabolic subalgebras, and then recall the techniques in [EHW], both of which offer us powerful tools to deal with our problem. In Section 3, we prove Theorem 1.1 in a case-by-case fashion for all seven cases.

2 Jantzen's Criterion and Special Lines

2.1 Jantzen's Criterion

In this section, we recall the irreducibility criterion due to J. C. Jantzen for generalized Verma modules. Because we focus on parabolic subalgebras of abelian type, which are automatically maximal parabolic subalgebras, we only state a specialization of Jantzen's criterion for scalar generalized Verma modules attached to maximal parabolic subalgebras \mathfrak{p} .

First of all, we fix some notations for the paper. All Lie algebras and modules considered in this paper are over the complex number field \mathbb{C} unless we make any declaration. Denote by $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ the set of nonnegative integers and the set of positive integers respectively. Let \mathfrak{g} be a finite-dimensional complex semisimple Lie algebra. Choose a Cartan subalgebra \mathfrak{h} of \mathfrak{g} , and a Borel subalgebra \mathfrak{b} containing \mathfrak{h} . Denote by Φ , Φ^+ , and Δ the root system, the set of positive roots, and the simple root system with respect to $(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ respectively. Let \mathfrak{p} be a maximal parabolic subalgebra such that $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$, and denote by $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ the Levi decomposition with respect to (\mathfrak{h}, Δ) , where \mathfrak{l} is the Levi factor with $\mathfrak{h} \subseteq \mathfrak{l}$ and $\mathfrak{l} + \mathfrak{b} = \mathfrak{p}$,

and \mathfrak{u}_+ is the nilpotent radical. Now we may denote by $\Phi_{\mathfrak{l}}$ the root system for $(\mathfrak{l}, \mathfrak{h})$, and set $\Phi_{\mathfrak{l}}^+ := \Phi_{\mathfrak{l}} \cap \Phi^+$ and $\Delta_{\mathfrak{l}} := \Phi_{\mathfrak{l}} \cap \Delta$. Also, define $\Phi_{\mathfrak{u}}^+ := \Phi^+ \setminus \Phi_{\mathfrak{l}}^+$. Let $(-, -)$ be the inner product on \mathfrak{h}^* induced by the Killing form of \mathfrak{g} , and for $\mu, \nu \in \mathfrak{h}^*$ define $\langle \mu, \nu \rangle := \frac{2(\mu, \nu)}{(\nu, \nu)}$. Now the set of $\Phi_{\mathfrak{l}}^+$ -dominant integral weights is defined as $\Lambda_{\mathfrak{l}}^+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Phi_{\mathfrak{l}}^+\}$. Moreover, denote by ρ half the sum of the positive roots in Φ^+ . For $\alpha \in \Phi$, define a reflection $s_{\alpha} : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ given by $s_{\alpha}(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha$ for $\lambda \in \mathfrak{h}^*$, and then denote by W (respectively, $W_{\mathfrak{l}}$) the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ (respectively, $(\mathfrak{l}, \mathfrak{h})$) generated by s_{α} for $\alpha \in \Delta$ (respectively, for $\alpha \in \Delta_{\mathfrak{l}}$). Let $U(\mathfrak{g})$ (respectively, $U(\mathfrak{p})$) be the universal enveloping algebra of \mathfrak{g} (respectively, \mathfrak{p}).

If $\lambda \in \mathfrak{h}^*$ is $\Phi_{\mathfrak{l}}^+$ -dominant integral, let F_{λ} be the finite-dimensional complex simple \mathfrak{l} -module with highest weight λ . By letting the elements in \mathfrak{u}_+ act by 0, F_{λ} is induced to a \mathfrak{p} -module. Now the generalized Verma module of \mathfrak{g} attached to \mathfrak{p} with the parameter λ is defined to be

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_{\lambda}.$$

When $\dim_{\mathbb{C}} F_{\lambda} = 1$, i.e., $(\lambda, \alpha) = 0$ for all $\alpha \in \Delta_{\mathfrak{l}}$, it is called a scalar generalized Verma module.

Theorem 2.1 ([H, Theorem 9.12]). *Let $\lambda \in \Lambda_{\mathfrak{l}}^+$. If $\langle \lambda + \rho, \beta \rangle \notin \mathbb{Z}_{>0}$ for all $\beta \in \Phi_{\mathfrak{u}}^+$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple. The converse also holds if $\lambda + \rho$ is regular.*

The following notation is important in the statement of Jantzen's criterion. For $\lambda \in \mathfrak{h}^*$, define

$$Y(\lambda) := D^{-1} \sum_{\omega \in W_{\mathfrak{l}}} (-1)^{l(\omega)} e^{\omega \lambda} \quad (2.1.1)$$

where $l(\omega)$ denotes the length of $\omega \in W_{\mathfrak{l}}$, e^{μ} is a function on \mathfrak{h}^* which takes values 1 at μ and 0 elsewhere, and $D = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})$ is the Weyl denominator. It is clear that $Y(\lambda)$ is the character formula of $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \rho)$ if λ is $\Phi_{\mathfrak{l}}^+$ -dominant integral.

Proposition 2.2 ([Ku, Corollary A.1.5], [M, Corollary 2.2.10]). *We have the following two properties:*

- (1) *If $\lambda \in \mathfrak{h}^*$ satisfies $(\lambda, \alpha) = 0$ for some $\alpha \in \Phi_{\mathfrak{l}}$, i.e., λ is $\Phi_{\mathfrak{l}}$ -singular, then $Y(\lambda) = 0$. Conversely, if $\lambda \in \mathfrak{h}^*$ satisfies $\langle \lambda, \alpha \rangle \in \mathbb{Z} \setminus \{0\}$ for all $\alpha \in \Phi_{\mathfrak{l}}$, i.e., λ is $\Phi_{\mathfrak{l}}$ -regular integral, then $Y(\lambda) \neq 0$.*
- (2) *For $\lambda \in \mathfrak{h}^*$ and $\omega \in W_{\mathfrak{l}}$, we have $Y(\omega \lambda) = (-1)^{l(\omega)} Y(\lambda)$.*

Denote by s_{β} the reflection corresponding to $\beta \in \Phi$ in W . Set

$$S_{\lambda} := \{\beta \in \Phi_{\mathfrak{u}}^+ \mid \langle \lambda + \rho, \beta \rangle \in \mathbb{Z}_{>0}\}. \quad (2.1.2)$$

Then Jantzen's criterion for the scalar generalized Verma modules attached to a maximal parabolic subalgebra \mathfrak{p} is stated as follows.

Theorem 2.3 ([J, Satz 3], [M, Theorem 2.2.11]). *Let \mathfrak{p} be a maximal parabolic subalgebra. Then the scalar generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is irreducible if and only if*

$$\sum_{\beta \in S_{\lambda}} Y(s_{\beta}(\lambda + \rho)) = 0.$$

To use Jantzen's criterion, we need to determine whether $\sum_{\beta \in S_\lambda} Y(s_\beta(\lambda + \rho))$ is 0 or not. This is answered by the following proposition.

Proposition 2.4 ([Ku, Proposition A.2.4]). *The sum $\sum_{\beta \in S_\lambda} Y(s_\beta(\lambda + \rho))$ is nonzero if and only if there is $\beta_0 \in S_\lambda$ satisfying the following two conditions: (a) $Y(s_{\beta_0}(\lambda + \rho)) \neq 0$; and (b) there do not exist $\beta \in S_\lambda \setminus \{\beta_0\}$ and $\omega \in W_I$ of odd length such that $s_{\beta_0}(\lambda + \rho) = \omega s_\beta(\lambda + \rho)$.*

2.2 Special Lines

Let us recall the construction of special lines in [EHW]. Henceforth \mathfrak{g} is a finite-dimensional complex simple Lie algebra, and \mathfrak{p} is a parabolic subalgebra of abelian type. Recall from Section 1 that $\zeta \in \mathfrak{h}^*$ is the unique weight such that $(\zeta, \alpha) = 0$ for all $\alpha \in \Phi_I$ and $\langle \zeta, \gamma \rangle = 1$.

Definition 2.5. If $\lambda_0 \in \mathfrak{h}^*$ satisfies $(\lambda_0 + \rho, \gamma) = 0$, then $\lambda_0 + z\zeta$ for $z \in \mathbb{C}$ is called a *special line* in \mathfrak{h}^* .

Obviously, every element $\lambda \in \mathfrak{h}^*$ can be expressed uniquely in the form $\lambda_0 + z\zeta$ with $z \in \mathbb{C}$ and $(\lambda_0 + \rho, \gamma) = 0$. Hence, every element $\lambda \in \mathfrak{h}^*$ lies in some special line. In fact, given $\lambda \in \mathfrak{h}^*$, if $\lambda = \lambda_0 + z\zeta$ with $z \in \mathbb{C}$ and $(\lambda_0 + \rho, \gamma) = 0$, then $z = \langle \lambda + \rho, \gamma \rangle$ and $\lambda_0 = \lambda - z\zeta$. This shows uniqueness. Moreover, $\langle \lambda_0 + \rho, \gamma \rangle = \langle \lambda - z\zeta + \rho, \gamma \rangle = \langle \lambda + \rho, \gamma \rangle - z = 0$, and existence is showed.

The following result can be found in [EHW], which we state as a theorem here.

Theorem 2.6. *For each special line $\lambda_0 + z\zeta$, there exist three real numbers $A(\lambda_0)$, $B(\lambda_0)$, and $C(\lambda_0)$ with $C(\lambda_0) > 0$, $A(\lambda_0) \leq B(\lambda_0)$, and $C(\lambda_0)^{-1}(B(\lambda_0) - A(\lambda_0)) \in \mathbb{Z}_{\geq 0}$ satisfying:*

- (1) *If $z \in \mathbb{R}$ and $z < A(\lambda_0)$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_0 + z\zeta)$ is simple.*
- (2) *If $z = A(\lambda_0) + iC(\lambda_0)$ for $0 \leq i \leq C(\lambda_0)^{-1}(B(\lambda_0) - A(\lambda_0))$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_0 + z\zeta)$ is reducible.*

Proof. See Theorem 2.4 and Proposition 3.1(b) in [EHW]. □

We shall make full use of the values of $A(\lambda_0)$, $B(\lambda_0)$, and $C(\lambda_0)$ listed in [EHW] to do computations for the Lie algebra pairs $(\mathfrak{g}, \mathfrak{p})$ with \mathfrak{g} simple and \mathfrak{p} a parabolic subalgebra of abelian type, case by case according to the Hermitian symmetric pairs.

3 Reducibility of Scalar Generalized Verma Modules of Abelian Type

In this section, we shall do computations case by case. According to [W], up to isomorphism there are seven complex Lie algebra pairs $(\mathfrak{g}, \mathfrak{p})$ with \mathfrak{g} simple and \mathfrak{p} a parabolic subalgebra of abelian type, which correspond to the Hermitian symmetric pairs of compact type:

- (i) $(SU(p+q), S(U(p) \times U(q)))$ for $p, q \geq 1$,
- (ii) $(Sp(n), U(n))$ for $n \geq 2$,
- (iii) $(SO(2n+1), SO(2) \times SO(2n-1))$ for $n \geq 2$,
- (iv) $(SO(2n), SO(2) \times SO(2n-2))$ for $n \geq 2$,
- (v) $(SO(2n), U(n))$ for $n \geq 2$,
- (vi) $(E_{6(-78)}, SO(2) \times SO(10))$,
- (vii) $(E_{7(-133)}, SO(2) \times E_{6(-78)})$.

In each case, it is easily verified that the Minkowski sum $\Phi_{\mathfrak{u}}^+ + \Phi_{\mathfrak{u}}^+ := \{\alpha + \beta \mid \alpha, \beta \in \Phi_{\mathfrak{u}}^+\} \subseteq \mathfrak{h}^*$ does not intersect $\Phi_{\mathfrak{u}}^+$, whence \mathfrak{u}_+ is abelian.

We see below that the computations for the first two pairs are almost trivial because we do not need to use Jantzen's criterion, while the computations for the last two pairs are more involved because the root structures and Weyl groups of exceptional types are complicated. Moreover, for the pair $(SO(2n), U(n))$, we have to discuss separately according to the parity of n . Retain all the notations and settings in the previous two sections.

3.1 $(SU(p+q), S(U(p) \times U(q)))$ for $p, q \geq 1$

Let $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of abelian type with $\mathfrak{l} = \mathfrak{s}(\mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C}))$. We may choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ and a simple root system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq p+q-1\}$, such that \mathfrak{p} is standard with respect to (\mathfrak{h}, Δ) . Then $\Delta_{\mathfrak{l}} = \{e_i - e_{i+1} \mid 1 \leq i \leq p+q-1, i \neq p\}$ and $\Phi_{\mathfrak{u}}^+ = \{e_i - e_j \mid i \leq p, j > p\}$. Moreover, we have $\rho = \sum_{i=1}^{p+q} \left(\frac{p+q+1}{2} - i\right) e_i$.

If $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of scalar type, an easy computation shows that $\lambda = \frac{aq}{p+q} \sum_{i=1}^p e_i - \frac{ap}{p+q} \sum_{j=p+1}^{p+q} e_j$ for some $a \in \mathbb{C}$.

Lemma 3.1. Suppose $\lambda = \frac{aq}{p+q} \sum_{i=1}^p e_i - \frac{ap}{p+q} \sum_{j=p+1}^{p+q} e_j$ for some $a \in \mathbb{C}$.

- (1) If $a \notin 2 - p - q + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $a \in \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. It is obvious that $\lambda + \rho$ is always $\Phi_{\mathfrak{l}}^+$ -dominant integral. For $i \leq p$ and $j > p$, one computes that $\langle \lambda + \rho, e_i - e_j \rangle = j - i + a$. Now the conclusion follows from Theorem 2.1 immediately. \square

The maximal root γ in Φ^+ is $e_1 - e_n$, and it follows that $\zeta = \frac{q}{p+q} \sum_{i=1}^p e_i - \frac{p}{p+q} \sum_{j=p+1}^{p+q} e_j$. If we write $\lambda = \lambda_0 + z\zeta$ in the special line, then we obtain $\lambda_0 = (\frac{q}{p+q} - q) \sum_{i=1}^p e_i + (p - \frac{p}{p+q}) \sum_{j=p+1}^{p+q} e_j$ and $a = z - p - q + 1$. Now we may restate Lemma 3.1.

Lemma 3.1'. Suppose $\lambda = \frac{aq}{p+q} \sum_{i=1}^p e_i + \frac{ap}{p+q} \sum_{j=p+1}^{p+q} e_j$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line.

- (1) If $z \notin 1 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $z \in p + q - 1 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Because $a = z - p - q + 1$, the conclusion follows from Lemma 3.1 immediately. \square

By Theorem 2.3, Lemma 7.3, and Theorem 7.4 in [EHW], one may check that $A(\lambda_0) = \max\{p, q\}$, $B(\lambda_0) = p + q - 1$, and $C(\lambda_0) = 1$ in this case.

Theorem 3.2. Suppose $\lambda = \frac{aq}{p+q} \sum_{i=1}^p e_i + \frac{ap}{p+q} \sum_{j=p+1}^{p+q} e_j$ for some $a \in \mathbb{C}$. Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $a \in 1 - \min\{p, q\} + \mathbb{Z}_{\geq 0}$.

Proof. Write $\lambda = \lambda_0 + z\zeta$ in the special line, and then Lemma 3.1'(1) and Theorem 2.6(1) show that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible only if $z \in \max\{p, q\} + \mathbb{Z}_{\geq 0}$. On the other hand, Lemma 3.1'(2) and Theorem 2.6(2) show that if $z \in \max\{p, q\} + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. It follows that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $z \in \max\{p, q\} + \mathbb{Z}_{\geq 0}$, which is equivalent to $a \in \max\{p, q\} - p - q + 1 + \mathbb{Z}_{\geq 0} = 1 - \min\{p, q\} + \mathbb{Z}_{\geq 0}$. \square

3.2 $(Sp(n), U(n))$ for $n \geq 2$

Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of abelian type with $\mathfrak{l} = \mathfrak{gl}(n, \mathbb{C})$. We may choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ and a simple root system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2e_n\}$, such that \mathfrak{p} is standard with respect to (\mathfrak{h}, Δ) . Then $\Delta_{\mathfrak{l}} = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$ and $\Phi_{\mathfrak{u}}^+ = \{e_i + e_j \mid 1 \leq i < j \leq n\} \cup \{2e_k \mid 1 \leq k \leq n\}$. Moreover, we have $\rho = \sum_{i=1}^n (n-i+1)e_i$.

If $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of scalar type, an easy computation shows that $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$.

Lemma 3.3. Suppose $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$.

- (1) If $a \notin 1 - n + \frac{1}{2}\mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.

(2) If $a \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. It is obvious that $\lambda + \rho$ is always $\Phi_{\mathfrak{l}}^+$ -dominant integral. For $1 \leq i < j \leq n$ and $1 \leq k \leq n$, one computes that $\langle \lambda + \rho, e_i + e_j \rangle = 2a + 2n - i - j + 2$ and $\langle \lambda + \rho, 2e_k \rangle = a + n - k + 1$. Now the conclusion follows from Theorem 2.1 immediately. \square

The maximal root γ in Φ^+ is $2e_1$, and it follows that $\zeta = \sum_{i=1}^n e_i$. If we write $\lambda = \lambda_0 + z\zeta$ in the special line, then we obtain $\lambda_0 = -n \sum_{i=1}^n e_i$ and $a = z - n$. Now we may restate Lemma 3.3.

Lemma 3.3'. Suppose $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line.

- (1) If $z \notin 1 + \frac{1}{2}\mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $z \in n + \frac{1}{2}\mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Because $a = z - n$, the conclusion follows from Lemma 3.3 immediately. \square

By Theorem 2.3, Lemma 8.3, and Theorem 8.4 in [EHW], one may check that $A(\lambda_0) = \frac{n+1}{2}$, $B(\lambda_0) = n$, and $C(\lambda_0) = \frac{1}{2}$ in this case. Unlike the case in [EHW, Section 7], it is not trivial to obtain these three numbers in this case. In fact, we have to show $q = r$ in Lemma 8.3 and Theorem 8.4 in [EHW]. According to the construction of $R(\lambda_0)$ and $Q(\lambda_0)$ under [EHW, Theorem 2.4], we have $R(\lambda_0) = Q(\lambda_0)$ which is the full root system of $\mathfrak{sp}(n)$, and then [EHW, Equation (8.1)] shows $q = r = n$ in our case. The three numbers $A(\lambda_0)$, $B(\lambda_0)$, and $C(\lambda_0)$ in the cases from Section 3.3 to Section 3.7 can be obtained by similar computations, but we shall only mention the three numbers and the details are left to the reader.

Theorem 3.4. Suppose $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$. Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $a \in \frac{1-n}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}$.

Proof. Write $\lambda = \lambda_0 + z\zeta$ in the special line, and then Lemma 3.3'(1) and Theorem 2.6(1) show that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible only if $z \in \frac{n+1}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}$. On the other hand, Lemma 3.3'(2) and Theorem 2.6(2) show that if $z \in \frac{n+1}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. It follows that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $z \in \frac{n+1}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}$, which is equivalent to $a \in \frac{1-n}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}$. \square

3.3 $(SO(2n+1), SO(2) \times SO(2n-1))$ for $n \geq 2$

Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of abelian type with $\mathfrak{l} = \mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{so}(2n-1, \mathbb{C})$. We may choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ and a simple root system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_n\}$, such that \mathfrak{p} is standard with respect to (\mathfrak{h}, Δ) . Then $\Delta_{\mathfrak{l}} = \{e_i - e_{i+1} \mid 2 \leq i \leq n-1\} \cup \{e_n\}$ and $\Phi_{\mathfrak{u}}^+ = \{e_1 \pm e_j \mid 2 \leq j \leq n\} \cup \{e_1\}$. Moreover, we have $\rho = \sum_{i=1}^n (n-i + \frac{1}{2})e_i$.

If $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of scalar type, an easy computation shows that $\lambda = ae_1$ for some $a \in \mathbb{C}$.

Lemma 3.5. Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$.

- (1) If $a \notin (3 - 2n + \mathbb{Z}_{\geq 0}) \cup (\frac{3}{2} - n + \mathbb{Z}_{\geq 0})$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $a \in \mathbb{Z}_{\geq 0} \cup (\frac{3}{2} - n + \mathbb{Z}_{\geq 0})$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. It is obvious that $\lambda + \rho$ is always Φ_{Γ}^+ -dominant integral. For $2 \leq j \leq n$, one computes that $\langle \lambda + \rho, e_1 + e_j \rangle = a + 2n - j$ and $\langle \lambda + \rho, e_1 - e_j \rangle = a + j - 1$. Moreover, $\langle \lambda + \rho, e_1 \rangle = 2a + 2n - 1$. Now the conclusion follows from Theorem 2.1 immediately. \square

The maximal root γ in Φ^+ is $e_1 + e_2$, and it follows that $\zeta = e_1$. If we write $\lambda = \lambda_0 + z\zeta$ in the special line, then we obtain $\lambda_0 = (2 - 2n)e_1$ and $a = z - 2n + 2$. Now we may restate Lemma 3.5.

Lemma 3.5'. Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line.

- (1) If $z \notin (1 + \mathbb{Z}_{\geq 0}) \cup (n - \frac{1}{2} + \mathbb{Z}_{\geq 0})$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $z \in (n - \frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (2n - 2 + \mathbb{Z}_{\geq 0})$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Because $a = z - 2n + 2$, the conclusion follows from Lemma 3.5 immediately. \square

By Theorem 2.3, Lemma 11.3, and Theorem 11.4 in [EHW], one may check that $A(\lambda_0) = n - \frac{1}{2}$, $B(\lambda_0) = 2n - 2$, and $C(\lambda_0) = n - \frac{3}{2}$ in this case.

Proposition 3.6. Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line. If $z \in \mathbb{Z}$ and $A(\lambda_0) < z < B(\lambda_0)$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.

Proof. Because $a = z - 2n + 2$, in fact we already computed that $\langle \lambda + \rho, e_1 + e_j \rangle = z - j + 2$, $\langle \lambda + \rho, e_1 - e_j \rangle = z - 2n + j + 1$ for $2 \leq j \leq n$, and $\langle \lambda + \rho, e_1 \rangle = 2z - 2n + 3$. If $z \in \mathbb{Z}$ and $A(\lambda_0) = n - \frac{1}{2} < z < B(\lambda_0) = 2n - 2$, then

$$S_{\lambda} = \{e_1 + e_j \mid 2 \leq j \leq n\} \cup \{e_1 - e_j \mid 2n - z - 1 < j \leq n\} \cup \{e_1\}.$$

Since $\lambda + \rho = (z - n + \frac{3}{2})e_1 + \sum_{i=2}^n (n - i + \frac{1}{2})e_i$, we have that $(s_{e_1 + e_j}(\lambda + \rho), e_{2n - z - 1} + e_j) = 0$ for $2 \leq j \leq n$ and $j \neq 2n - z - 1$. It follows if $2 \leq j \leq n$ and $j \neq 2n - z - 1$, then $s_{e_1 + e_j}(\lambda + \rho)$ is Φ_{Γ} -singular, and hence $Y(s_{e_1 + e_j}(\lambda + \rho)) = 0$ by Proposition 2.2(1). Similarly, we have $(s_{e_1 - e_j}(\lambda + \rho), e_{2n - z - 1} - e_j) = 0$ for $2n - z - 1 < j \leq n$, so $Y(s_{e_1 - e_j}(\lambda + \rho)) = 0$ for $2n - z - 1 < j \leq n$. On the other hand, it is easy to check that $s_{e_1 + e_{2n - z - 1}}(\lambda + \rho)$ and $s_{e_1}(\lambda + \rho)$ are Φ_{Γ} -regular integral, and hence $Y(s_{e_1 + e_{2n - z - 1}}(\lambda + \rho)) \neq 0$ and $Y(s_{e_1}(\lambda + \rho)) \neq 0$ by Proposition 2.2(1). Moreover, because $s_{e_1 + e_{2n - z - 1}}(\lambda + \rho) = s_{e_{2n - z - 1}}s_{e_1}(\lambda + \rho)$ and $s_{e_{2n - z - 1}} \in W_{\Gamma}$ is of odd length, it follows from Proposition 2.2(2) that $Y(s_{e_1 + e_{2n - z - 1}}(\lambda + \rho)) + Y(s_{e_1}(\lambda + \rho)) = 0$. Now

$$\sum_{\beta \in S_{\lambda}} Y(s_{\beta}(\lambda + \rho)) = Y(s_{e_1 + e_{2n - z - 1}}(\lambda + \rho)) + Y(s_{e_1}(\lambda + \rho)) = 0.$$

By Theorem 2.3, $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple. \square

Theorem 3.7. Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$. Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $a \in \mathbb{Z}_{\geq 0} \cup (\frac{3}{2} - n + \mathbb{Z}_{\geq 0})$.

Proof. Write $\lambda = \lambda_0 + z\zeta$ in the special line, and then Lemma 3.5'(1), Proposition 3.6, and Theorem 2.6(1) show that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible only if $z \in (n - \frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (2n - 2 + \mathbb{Z}_{\geq 0})$. On the other hand, Lemma 3.5'(2) shows that if $z \in (n - \frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (2n - 2 + \mathbb{Z}_{\geq 0})$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. It follows that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $z \in (n - \frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup (2n - 2 + \mathbb{Z}_{\geq 0})$, which is equivalent to $a \in \mathbb{Z}_{\geq 0} \cup (\frac{3}{2} - n + \mathbb{Z}_{\geq 0})$. \square

3.4 $(SO(2n), SO(2) \times SO(2n - 2))$ for $n \geq 2$

Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of abelian type with $\mathfrak{l} = \mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{so}(2n - 2, \mathbb{C})$. We may choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ and a simple root system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{e_{n-1} + e_n\}$, such that \mathfrak{p} is standard with respect to (\mathfrak{h}, Δ) . Then $\Delta_{\mathfrak{l}} = \{e_i - e_{i+1} \mid 2 \leq i \leq n - 1\} \cup \{e_{n-1} + e_n\}$ and $\Phi_{\mathfrak{u}}^+ = \{e_1 \pm e_j \mid 2 \leq j \leq n\}$. Moreover, we have $\rho = \sum_{i=1}^n (n - i)e_i$.

If $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of scalar type, an easy computation shows that $\lambda = ae_1$ for some $a \in \mathbb{C}$.

Lemma 3.8. Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$.

- (1) If $a \notin 4 - 2n + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $a \in \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. It is obvious that $\lambda + \rho$ is always $\Phi_{\mathfrak{l}}^+$ -dominant integral. For $2 \leq j \leq n$, one computes that $\langle \lambda + \rho, e_1 + e_j \rangle = a + 2n - j - 1$ and $\langle \lambda + \rho, e_1 - e_j \rangle = a + j - 1$. Now the conclusion follows from Theorem 2.1 immediately. \square

The maximal root γ in Φ^+ is $e_1 + e_2$, and it follows that $\zeta = e_1$. If we write $\lambda = \lambda_0 + z\zeta$ in the special line, then we obtain $\lambda_0 = (3 - 2n)e_1$ and $a = z - 2n + 3$. Now we may restate Lemma 3.8.

Lemma 3.8'. Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line.

- (1) If $z \notin 1 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $z \in 2n - 3 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Because $a = z - 2n + 3$, the conclusion follows from Lemma 3.8 immediately. \square

By Theorem 2.3, Lemma 10.3, and Theorem 10.4 in [EHW], one may check that $A(\lambda_0) = n - 1$, $B(\lambda_0) = 2n - 3$, and $C(\lambda_0) = n - 2$ in this case.

Proposition 3.9. *Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line. If $z \in \mathbb{Z}$ and $A(\lambda_0) < z < B(\lambda_0)$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.*

Proof. Because $a = z - 2n + 3$, in fact we already computed that $\langle \lambda + \rho, e_1 + e_j \rangle = z - j + 2$ and $\langle \lambda + \rho, e_1 - e_j \rangle = z - 2n + j + 2$ for $2 \leq j \leq n$. If $z \in \mathbb{Z}$ and $A(\lambda_0) = n - 1 < z < B(\lambda_0) = 2n - 3$, then

$$S_{\lambda} = \{e_1 + e_j \mid 2 \leq j \leq n\} \cup \{e_1 - e_j \mid 2n - z - 2 < j \leq n\}.$$

Since $\lambda + \rho = (z - n + 2)e_1 + \sum_{i=2}^n (n - i)e_i$, we have that $(s_{e_1 + e_j}(\lambda + \rho), e_{2n - z - 2} + e_j) = 0$ for $2 \leq j \leq n$ and $j \neq 2n - z - 2$. It follows that if $2 \leq j \leq n$ and $j \neq 2n - z - 2$, then $s_{e_1 + e_j}(\lambda + \rho)$ is $\Phi_{\mathfrak{l}}$ -singular, and hence $Y(s_{e_1 + e_j}(\lambda + \rho)) = 0$ by Proposition 2.2(1). Similarly, if $2n - z - 2 < j \leq n$, then $(s_{e_1 - e_j}(\lambda + \rho), e_{2n - z - 2} - e_j) = 0$, and hence $Y(s_{e_1 - e_j}(\lambda + \rho)) = 0$. On the other hand, it is easy to check that $s_{e_1 + e_{2n - z - 2}}(\lambda + \rho)$ is $\Phi_{\mathfrak{l}}$ -regular integral, and hence $Y(s_{e_1 + e_{2n - z - 2}}(\lambda + \rho)) \neq 0$ by Proposition 2.2(1). Now

$$\sum_{\beta \in S_{\lambda}} Y(s_{\beta}(\lambda + \rho)) = Y(s_{e_1 + e_{2n - z - 2}}(\lambda + \rho)) \neq 0.$$

By Theorem 2.3, $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. \square

Theorem 3.10. *Suppose $\lambda = ae_1$ for some $a \in \mathbb{C}$. Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $a \in 2 - n + \mathbb{Z}_{\geq 0}$.*

Proof. Write $\lambda = \lambda_0 + z\zeta$ in the special line, and then Lemma 3.8'(1) and Theorem 2.6(1) show that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible only if $z \in n - 1 + \mathbb{Z}_{\geq 0}$. On the other hand, Lemma 3.8'(2), Proposition 3.9, and Theorem 2.6(2) show that if $z \in n - 1 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. It follows that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $z \in n - 1 + \mathbb{Z}_{\geq 0}$, which is equivalent to $a \in 2 - n + \mathbb{Z}_{\geq 0}$. \square

3.5 $(SO(2n), U(n))$ for $n \geq 2$

Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of abelian type with $\mathfrak{l} = \mathfrak{gl}(n, \mathbb{C})$. We may choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ and a simple root system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{e_{n-1} + e_n\}$, such that \mathfrak{p} is standard with respect to (\mathfrak{h}, Δ) . Then $\Delta_{\mathfrak{l}} = \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\}$ and $\Phi_{\mathfrak{u}}^+ = \{e_i + e_j \mid 1 \leq i < j \leq n\}$. Moreover, we have $\rho = \sum_{i=1}^n (n - i)e_i$.

If $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of scalar type, an easy computation shows that $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$.

Lemma 3.11. Suppose $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$.

(1) If $a \notin 2 - n + \frac{1}{2}\mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.

(2) If $a \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. It is obvious that $\lambda + \rho$ is always Φ_{Γ}^+ -dominant integral. For $1 \leq i < j \leq n$, one computes that $\langle \lambda + \rho, e_i + e_j \rangle = 2a + 2n - i - j$. Now the conclusion follows from Theorem 2.1 immediately. \square

The maximal root γ in Φ^+ is $e_1 + e_2$, and it follows that $\zeta = \frac{1}{2} \sum_{i=1}^n e_i$. If we write $\lambda = \lambda_0 + z\zeta$ in the special line, then we obtain $\lambda_0 = (\frac{3}{2} - n) \sum_{i=1}^n e_i$ and $z = 2a + 2n - 3$. Now we may restate Lemma 3.11.

Lemma 3.11'. Suppose $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line.

- (1) If $z \notin 1 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $z \in 2n - 3 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Because $z = 2a + 2n - 3$, the conclusion follows from Lemma 3.11 immediately. \square

By Theorem 2.3, Lemma 9.3, and Theorem 9.4 in [EHW], one may check that $B(\lambda_0) = 2n - 3$ and $C(\lambda_0) = 2$ in this case. The value of $A(\lambda_0)$ depends on the parity of n . If n is even, and $A(\lambda_0) = n - 1$; and if n is odd, $A(\lambda_0) = n$.

Proposition 3.12. Suppose $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line. If $z \in \mathbb{Z}$ and $A(\lambda_0) < z < B(\lambda_0)$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Let us assume that n is even, and then $A(\lambda_0) = n - 1$. Therefore Proposition 3.1(b), Lemma 9.3, and Theorem 9.4 in [EHW] imply that if $z = A(\lambda_0) + 2k$ for $k \in \mathbb{Z}_{\geq 0}$ and $z \leq B(\lambda_0)$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. Thus we only need to check $z = A(\lambda_0) + 2k + 1 = n + 2k$ for $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq \frac{n}{2} - 2$.

Because $z = 2a + 2n - 3$, in fact we already computed that $\langle \lambda + \rho, e_i + e_j \rangle = n + 2k - i - j + 3$ for $1 \leq i < j \leq n$. If $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq \frac{n}{2} - 2$, then

$$S_{\lambda} = \{e_i + e_j \mid i + j < n + 2k + 3\}.$$

Now consider $\lambda + \rho = \sum_{m=1}^n (\frac{n+3}{2} + k - m)e_m$. If $j \geq 2k + 3$, since $i + j < n + 2k + 3$, then $(s_{e_i+e_j}(\lambda + \rho), e_i - e_{n+2k+3-j}) = 0$. It follows that $s_{e_i+e_j}(\lambda + \rho)$ is Φ_{Γ} -singular, so $Y(s_{e_i+e_j}(\lambda + \rho)) = 0$ by Proposition 2.2(1). Hence, we only need to consider $Y(s_{e_i+e_j}(\lambda + \rho))$ for $i < j < 2k + 3$. It is obvious that $s_{e_1+e_2}(\lambda + \rho)$ is Φ_{Γ} -regular integral, and then $Y(s_{e_1+e_2}(\lambda + \rho)) \neq 0$ by Proposition 2.2(1). For $i < j < 2k + 3$ and $e_i + e_j \neq e_1 + e_2$, we claim that if there exist $e_i + e_j \in S_{\lambda}$ and $\omega \in W_{\Gamma}$ such that $s_{e_1+e_2}(\lambda + \rho) = \omega s_{e_i+e_j}(\lambda + \rho)$, then $i = 1$ and $j = 2$, so $\sum_{\beta \in S_{\lambda}} Y(s_{\beta}(\lambda + \rho)) \neq 0$ by Proposition 2.4, and hence the conclusion holds by

Theorem 2.3. To show this claim, let $c_m := \frac{n+3}{2} + k - m$ denote the coefficient of e_m in $\lambda + \rho$, i.e., let $\lambda + \rho = \sum_{m=1}^n c_m e_m$. It is easy to see that $c_1 > c_2 > |c_m|$ for $3 \leq m \leq n$, and

$s_{e_1+e_2}(\lambda+\rho) = -c_2e_1 - c_1e_2 + \sum_{m=3}^n c_me_m$. Because $W_l \cong S_n$ is a symmetric group, each $\omega \in W_l$ is a permutation of the basis vectors e_m for $1 \leq m \leq n$. It follows that $i = 1$ and $j = 2$ if $s_{e_1+e_2}(\lambda+\rho) = \omega s_{e_i+e_j}(\lambda+\rho)$ for some $\omega \in W_l$, since $-c_1$ and $-c_2$ must appear as coefficients of two basis vectors in $s_{e_i+e_j}(\lambda+\rho)$ and $-c_1 < -c_2 < -|c_m|$ for $3 \leq m \leq n$. The claim holds, and the conclusion for n even is proved.

The proof for n odd is parallel. If n is odd, then $A(\lambda_0) = n$, and we need to check $z = n+2k+1$ for $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq \frac{n-5}{2}$. It is computed that

$$S_\lambda = \{e_i + e_j \mid i + j < n + 2k + 4\}.$$

Finally, verify similarly to the above analysis that $Y(s_{e_1+e_2}(\lambda+\rho)) \neq 0$ cannot be cancelled out in $\sum_{\beta \in S_\lambda} Y(s_\beta(\lambda+\rho))$, and apply Theorem 2.3 to conclude the proof. \square

Theorem 3.13. *Suppose $\lambda = a \sum_{i=1}^n e_i$ for some $a \in \mathbb{C}$. Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if*

$$a \in \begin{cases} 1 - \frac{n}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}, & n \text{ even}, \\ \frac{3-n}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}, & n \text{ odd}. \end{cases}$$

Proof. Assume that n is even. Write $\lambda = \lambda_0 + z\zeta$ in the special line, and then Lemma 3.11'(1) and Theorem 2.6(1) show that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible only if $z \in n-1 + \mathbb{Z}_{\geq 0}$. On the other hand, Lemma 3.11'(2), Proposition 3.12, and Theorem 2.6(2) show that if $z \in n-1 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. It follows that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $z \in n-1 + \mathbb{Z}_{\geq 0}$, which is equivalent to $a \in 1 - \frac{n}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}$. The proof for n odd is parallel. \square

3.6 $(E_{6(-78)}, SO(2) \times SO(10))$

Let $\mathfrak{g} = \mathfrak{e}_6$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of abelian type with $\mathfrak{l} = \mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{so}(10, \mathbb{C})$. We may choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ and a simple root system $\Delta = \{\alpha_i \mid 1 \leq i \leq 6\}$ given by the Dynkin diagram of Figure 1 such that \mathfrak{p} is standard with respect to (\mathfrak{h}, Δ) . Then $\Delta_{\mathfrak{l}} = \{\alpha_i \mid 2 \leq i \leq 6\}$.

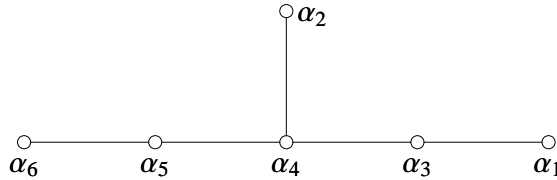


Figure 1: Dynkin diagram of \mathfrak{e}_6 .

Embed $\mathfrak{h}_{\mathbb{R}}^*$, the \mathbb{R} -span of the simple roots, into the subspace $V_6 := \{v \in \mathbb{R}^8 \mid (v, e_6 - e_7) = (v, e_7 + e_8) = 0\}$ of \mathbb{R}^8 , and let $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$, $\alpha_2 = e_1 + e_2$,

$\alpha_i = e_{i-1} - e_{i-2}$ for $3 \leq i \leq 6$. Now given $v \in \mathbb{Z}^5$, define

$$\alpha_{\pm}(v) := \frac{1}{2} \left(\sum_{i=1}^5 (-1)^{v(i)} e_i \pm e_6 - e_7 + e_8 \right). \quad (3.6.1)$$

Then

$$\Phi_l^+ = \{e_j \pm e_i \mid 1 \leq i < j \leq 5\}$$

and

$$\Phi_u^+ = \{\alpha_-(v) \mid \sum_{i=1}^5 v(i) \text{ even}\}.$$

Moreover, we have $\rho = e_2 + 2e_3 + 3e_4 + 4e_5 - 4e_6 - 4e_7 + 4e_8$.

If $M_p^g(\lambda)$ is of scalar type, an easy computation shows that $\lambda = -ae_6 - ae_7 + ae_8$ for some $a \in \mathbb{C}$.

Lemma 3.14. Suppose $\lambda = -ae_6 - ae_7 + ae_8$ for some $a \in \mathbb{C}$.

- (1) If $a \notin -\frac{20}{3} + \frac{2}{3}\mathbb{Z}_{\geq 0}$, then $M_p^g(\lambda)$ is simple.
- (2) If $a \in \frac{2}{3}\mathbb{Z}_{\geq 0}$, then $M_p^g(\lambda)$ is reducible.

Proof. It is obvious that $\lambda + \rho$ is always Φ_l^+ -dominant integral. For $\alpha_-(v) \in \Phi_u^+$, one computes that $\langle \lambda + \rho, \alpha_-(v) \rangle = \frac{1}{2}(3a + (-1)^{v(2)} + 2(-1)^{v(3)} + 3(-1)^{v(4)} + 4(-1)^{v(5)}) + 6$. Now the conclusion follows from Theorem 2.1 immediately. \square

The maximal root γ in Φ^+ is $\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)$, and it follows that $\zeta = \frac{2}{3}(-e_6 - e_7 + e_8)$. If we write $\lambda = \lambda_0 + z\zeta$ in the special line, then we obtain $\lambda_0 = \frac{22}{3}e_6 + \frac{22}{3}e_7 - \frac{22}{3}e_8$ and $a = \frac{2z-22}{3}$. Now we may restate Lemma 3.14.

Lemma 3.14'. Suppose $\lambda = -ae_6 - ae_7 + ae_8$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line.

- (1) If $z \notin 1 + \mathbb{Z}_{\geq 0}$, then $M_p^g(\lambda)$ is simple.
- (2) If $z \in 1 + \mathbb{Z}_{\geq 0}$, then $M_p^g(\lambda)$ is reducible.

Proof. Because $a = \frac{2z-22}{3}$, the conclusion follows from Lemma 3.14 immediately. \square

By Theorem 2.3, Lemma 12.3, and Theorem 12.4 in [EHW], one may check that $A(\lambda_0) = 8$, $B(\lambda_0) = 11$, and $C(\lambda_0) = 3$ in this case.

Proposition 3.15. Suppose $\lambda = -ae_6 - ae_7 + ae_8$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line. If $z = 9$, then $M_p^g(\lambda)$ is reducible.

Proof. If $z = 9$, then $a = -\frac{4}{3}$. For $\alpha_-(v) \in \Phi_u^+$, we have

$$\langle \lambda + \rho, \alpha_-(v) \rangle = \frac{1}{2}((-1)^{v(2)} + 2(-1)^{v(3)} + 3(-1)^{v(4)} + 4(-1)^{v(5)}) + 4.$$

Table 1: Coefficients of e_i for $1 \leq i \leq 5$ of $s_\beta(\lambda + \rho)$ for $\beta \in S_\lambda$ with $z = 9$

β	e_1	e_2	e_3	e_4	e_5
+++++	$-\frac{9}{2}$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$
--+++	4	5	-2	-1	0
-+-++	$\frac{7}{2}$	$-\frac{5}{2}$	$\frac{11}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
-++-+	3	-2	-1	6	1
-++++	$\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{13}{2}$
+---+	-3	4	5	0	1
+--+-	$-\frac{5}{2}$	$\frac{7}{2}$	$-\frac{1}{2}$	$\frac{11}{2}$	$\frac{3}{2}$
+---+	-2	3	0	1	6
++--+	-2	-1	4	5	2
++-+-	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{11}{2}$
+++--	-1	0	1	4	5
---++	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{5}{2}$
---+-	1	2	3	2	5
---+-	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{9}{2}$

Then

$$S_\lambda = \{\alpha_-(v) \in \Phi_u^+ \mid \text{at least one of } v(3), v(4), \text{ and } v(5) \text{ is even}\}.$$

We need to compute $s_\beta(\lambda + \rho)$. Here $\lambda + \rho = e_2 + 2e_3 + 3e_4 + 4e_5 + (-a-4)e_6 + (-a-4)e_7 + (a+4)e_8$. We do not need to work out all the coefficients, but only the coefficients of e_i for $1 \leq i \leq 5$. Moreover, we use five “+” and “-” symbols to indicate the parity of $v(i)$ for $1 \leq i \leq 5$ in β , where “+” corresponds to $v(i)$ even and “-” corresponds to $v(i)$ odd. For example, “++-+-” represents $\frac{1}{2}(e_1 + e_2 - e_3 - e_4 + e_5 - e_6 - e_7 + e_8)$, and “-----” represents $\frac{1}{2}(-e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$.

According to Table 1, it is immediate that $s_\beta(\lambda + \rho)$ is Φ_l -regular integral precisely when β is represented by “+++++”, “--+++”, “+-++-”, “+--+-” or “+-++-”. Because the elements in W_l do not change the parity of the number of positive coefficients and do not change the number of zeros, it follows that there do not exist $\omega \in W_l$ and β not represented by “+++++” such that $s_\beta(\lambda + \rho) = \omega s_{\beta_0}(\lambda + \rho)$ for β_0 represented by “+++++”. Therefore, $\sum_{\beta \in S_\lambda} Y(s_\beta(\lambda + \rho)) \neq 0$ by Proposition 2.4, and hence the conclusion holds by Theorem 2.3. \square

Proposition 3.16. *Suppose $\lambda = -ae_6 - ae_7 + ae_8$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line. If $z = 10$, then $M_p^g(\lambda)$ is reducible.*

Proof. If $z = 10$, then $a = -\frac{2}{3}$. For $\alpha_-(v) \in \Phi_u^+$, we have

$$\langle \lambda + \rho, \alpha_-(v) \rangle = \frac{1}{2}((-1)^{v(2)} + 2(-1)^{v(3)} + 3(-1)^{v(4)} + 4(-1)^{v(5)}) + 5.$$

Then

$$S_\lambda = \{\alpha_-(v) \in \Phi_u^+ \mid \text{at least one of } v(2), v(3), v(4), \text{ and } v(5) \text{ is even}\}.$$

Similar to Proposition 3.15, we need to compute the coefficients of e_i for $1 \leq i \leq 5$ of $s_\beta(\lambda + \rho)$.

Table 2: Coefficients of e_i for $1 \leq i \leq 5$ of $s_\beta(\lambda + \rho)$ for $\beta \in S_\lambda$ with $z = 10$

β	e_1	e_2	e_3	e_4	e_5
+++++	-5	-4	-3	-2	-1
--+++	$\frac{9}{2}$	$\frac{11}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$
-+-++	4	-3	6	-1	0
-++-+	$\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$\frac{13}{2}$	$\frac{1}{2}$
-++++	3	-2	-1	0	7
+---+	$-\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
+--+-	-3	4	-1	6	1
+---+	$-\frac{5}{2}$	$\frac{7}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{13}{2}$
++--+	$-\frac{5}{2}$	$-\frac{3}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{3}{2}$
++-+-	-2	-1	4	1	6
+++--	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{9}{2}$	$\frac{11}{2}$
----+	2	3	4	5	2
---+-	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{11}{2}$
---+-	1	2	1	4	5
-+---	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$

According to Table 2, it is immediate that $s_\beta(\lambda + \rho)$ is Φ_Γ -regular precisely when β is represented by “++++”, “--+++”, “-+-++”, “-++++” or “-+++-”. For the same reason as in Proposition 3.15, $Y(s_{\beta_0}(\lambda + \rho))$ for β_0 represented by “-+++-” leads to $\sum_{\beta \in S_\lambda} Y(s_\beta(\lambda + \rho)) \neq 0$ by Proposition 2.4, and hence the conclusion holds by Theorem 2.3. \square

Theorem 3.17. *Suppose $\lambda = -ae_6 - ae_7 + ae_8$ for some $a \in \mathbb{C}$. Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $a \in -2 + \frac{2}{3}\mathbb{Z}_{\geq 0}$.*

Proof. Write $\lambda = \lambda_0 + z\zeta$ in the special line, and then Lemma 3.14'(1) and Theorem 2.6(1) show that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible only if $z \in 8 + \mathbb{Z}_{\geq 0}$. On the other hand, Lemma 3.14'(2), Proposition 3.15, Proposition 3.16, and Theorem 2.6(2) show that if $z \in 8 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. It follows that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $z \in 8 + \mathbb{Z}_{\geq 0}$, which is equivalent to $a \in -2 + \frac{2}{3}\mathbb{Z}_{\geq 0}$. \square

3.7 $(E_{7(-133)}, SO(2) \times E_{6(-78)})$

Let $\mathfrak{g} = \mathfrak{e}_7$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a parabolic subalgebra of abelian type with $\mathfrak{l} = \mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{e}_6$. We may choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ and a simple root system $\Delta = \{\alpha_i \mid 1 \leq i \leq 7\}$ given by the Dynkin diagram of Figure 2 such that \mathfrak{p} is standard with respect to (\mathfrak{h}, Δ) . Then $\Delta_{\mathfrak{l}} = \{\alpha_i \mid 1 \leq i \leq 6\}$ of Δ .

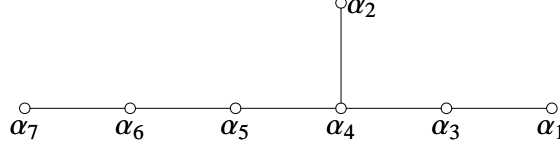


Figure 2: Dynkin diagram of \mathfrak{e}_7 .

Embed $\mathfrak{h}_{\mathbb{R}}^*$, the \mathbb{R} -span of the simple roots, into the subspace $V_7 := \{v \in \mathbb{R}^8 \mid (v, e_7 + e_8) = 0\}$ of \mathbb{R}^8 . Then Φ_1^+ equals the full set of the positive roots of \mathfrak{e}_6 as in Section 3.6, while $\alpha_7 = e_6 - e_5$ and

$$\Phi_u^+ = \{e_6 \pm e_i \mid 1 \leq i \leq 5\} \cup \{e_8 - e_7\} \cup \{\alpha_+(v) \mid \sum_{i=1}^5 v(i) \text{ odd}\}.$$

Here we retain the notations $\alpha_{\pm}(v)$ as in (3.6.1). Moreover, we have $\rho = e_2 + 2e_3 + 3e_4 + 4e_5 + 5e_6 - \frac{17}{2}e_7 + \frac{17}{2}e_8$.

If $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of scalar type, an easy computation shows that $\lambda = ae_6 - \frac{1}{2}ae_7 + \frac{1}{2}ae_8$ for some $a \in \mathbb{C}$.

Lemma 3.18. Suppose $\lambda = ae_6 - \frac{1}{2}ae_7 + \frac{1}{2}ae_8$ for some $a \in \mathbb{C}$.

- (1) If $a \notin -16 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $a \in \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. It is obvious that $\lambda + \rho$ is always Φ_1^+ -dominant integral. Firstly, $\langle \lambda + \rho, e_8 - e_7 \rangle = a + 17$. Secondly, $\langle \lambda + \rho, e_6 \pm e_i \rangle = a + 5 \pm (i - 1)$ for $1 \leq i \leq 5$. Thirdly, for $\alpha_+(v) \in \Phi_u^+$, one computes that $\langle \lambda + \rho, \alpha_+(v) \rangle = a + 11 + \frac{1}{2}((-1)^{v(2)} + 2(-1)^{v(3)} + 3(-1)^{v(4)} + 4(-1)^{v(5)})$. Now the conclusion follows from Theorem 2.1 immediately. \square

The maximal root γ in Φ^+ is $e_8 - e_7$, and it follows that $\zeta = e_6 - \frac{1}{2}e_7 + \frac{1}{2}e_8$. If we write $\lambda = \lambda_0 + z\zeta$ in the special line, then we obtain $\lambda_0 = -17e_6 + \frac{17}{2}e_7 - \frac{17}{2}e_8$ and $a = z - 17$. Now we may restate Lemma 3.18.

Lemma 3.18'. Suppose $\lambda = ae_6 - \frac{1}{2}ae_7 + \frac{1}{2}ae_8$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line.

- (1) If $z \notin 1 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple.
- (2) If $z \in 17 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Because $a = z - 17$, the conclusion follows from Lemma 3.18 immediately. \square

By Theorem 2.3, Lemma 13.3, and Theorem 13.4 in [EHW], one may check that $A(\lambda_0) = 9$, $B(\lambda_0) = 17$, and $C(\lambda_0) = 4$ in this case.

Actually, we only need to check $z \in \{10, 11, 12, 14, 15, 16\}$. However, the Weyl group of W_I is too complicated. It is hard to verify whether $\sum_{\beta \in S_\lambda} Y(s_\beta(\lambda + \rho))$ equals 0 by means of Proposition 2.4. Therefore, we provide an alternate method to solve the problem for \mathfrak{e}_7 .

Lemma 3.19. Let α_u be the unique simple root in Φ_u^+ , and denote by θ_u the fundamental weight of α_u . Suppose $\mu, \nu \in \mathfrak{h}^*$. If $\mu = \omega\nu$ for some $\omega \in W_I$, then $(\mu, \theta_u) = (\nu, \theta_u)$.

Proof. Because $\omega \in W_I$, $\omega\alpha \in \Phi_I$ for all $\alpha \in \Delta_I$. Hence $\langle \omega\theta_u, \alpha \rangle = \frac{2(\omega\theta_u, \alpha)}{(\alpha, \alpha)} = \frac{2(\theta_u, \omega^{-1}\alpha)}{(\alpha, \alpha)} = 0$ for all $\alpha \in \Delta_I$. On the other hand, $\omega^{-1}\alpha_u = \alpha_u + \beta$ for some $\beta \in \text{span}_{\mathbb{Z}}\Delta_I$, and it follows that

$$\langle \omega\theta_u, \alpha_u \rangle = \frac{2(\omega\theta_u, \alpha_u)}{(\alpha_u, \alpha_u)} = \frac{2(\theta_u, \omega^{-1}\alpha_u)}{(\alpha_u, \alpha_u)} = \frac{2(\theta_u, \alpha_u + \beta)}{(\alpha_u, \alpha_u)} = \frac{2(\theta_u, \alpha_u)}{(\alpha_u, \alpha_u)} = \langle \theta_u, \alpha_u \rangle = 1.$$

Therefore, $\omega\theta_u = \theta_u$. Now if $\mu = \omega\nu$ for some $\omega \in W_I$, then $(\mu, \theta_u) = (\omega\nu, \theta_u) = (\nu, \omega^{-1}\theta_u) = (\nu, \theta_u)$. \square

Proposition 3.20. Suppose $\lambda = ae_6 - \frac{1}{2}ae_7 + \frac{1}{2}ae_8$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line. If $z \in \{12, 14, 15, 16\}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. The only simple root in Φ_u^+ is $\alpha_7 = e_6 - e_5$, and the fundamental weight θ_7 of α_7 is $e_6 - \frac{1}{2}e_7 + \frac{1}{2}e_8$. It is easy to write down that $\lambda + \rho = e_2 + 2e_3 + 3e_4 + 4e_5 + (a+5)e_6 - \frac{a+17}{2}e_7 + \frac{a+17}{2}e_8$. In order to apply Proposition 2.4 via the contrapositive of Lemma 3.19, we need to compute $(s_\beta(\lambda + \rho), \theta_7)$ for $\beta \in \Phi_u^+$. It is almost immediate to work out that

$$(s_{e_6 \pm e_i}(\lambda + \rho), \theta_7) = \frac{a+17}{2} \pm (i-1)$$

for $1 \leq i \leq 5$ and

$$(s_{e_8 - e_7}(\lambda + \rho), \theta_7) = \frac{a-7}{2}.$$

As for the roots in $\{\alpha_+(v) \mid \sum_{i=1}^5 v(i) \text{ odd}\}$, let us still use the notation as in Proposition 3.15.

Because θ_7 only involves e_6, e_7 , and e_8 , we just write down the coefficients of these three vectors for each $s_\beta(\lambda + \rho)$. The values are listed in Table 3. According to Table 3, it is immediate that

$$(s_{e_8 - e_7}(\lambda + \rho), \theta_7) = \frac{a-7}{2} \neq (s_\beta(\lambda + \rho), \theta_7)$$

for all $\beta \in \{\alpha_+(v) \mid \sum_{i=1}^5 v(i) \text{ odd}\}$. Moreover,

$$(s_{e_8 - e_7}(\lambda + \rho), \theta_7) = \frac{a-7}{2} \neq \frac{a+17}{2} \pm (i-1)$$

for $1 \leq i \leq 5$. Hence by Lemma 3.19, there does not exist $\omega \in W_I$ such that $\omega s_{e_8 - e_7}(\lambda + \rho)$ equals $s_\beta(\lambda + \rho)$ for any other $\beta \in \Phi_u^+$. On the other hand, if $z \in \{12, 14, 15, 16\}$, then $\langle \lambda + \rho, e_8 - e_7 \rangle = a + 17 = z \in \mathbb{Z}_{>0}$, so $e_8 - e_7 \in S_\lambda$.

What remains to prove is that $Y(s_{e_8 - e_7}(\lambda + \rho)) \neq 0$, and $Y(s_{e_8 - e_7}(\lambda + \rho))$ leads to $\sum_{\beta \in S_\lambda} Y(s_\beta(\lambda + \rho)) \neq$

0 by Proposition 2.4, so the conclusion follows from Theorem 2.3. In fact, it is obvious that $(s_{e_8 - e_7}(\lambda + \rho), e_j \pm e_i) = j - i \in \mathbb{Z} \setminus \{0\}$ for $1 \leq i < j \leq 5$. For $\alpha = \alpha_-(v) \in \Phi_I^+$, we have

$$(s_{e_8 - e_7}(\lambda + \rho), \alpha) = \frac{1}{2}(-1)^{v(2)} + (-1)^{v(3)} + \frac{3}{2}(-1)^{v(4)} + 2(-1)^{v(5)} - 11 - a.$$

Table 3: Coefficients of e_6 , e_7 , and e_8 of $s_\beta(\lambda + \rho)$ and $(s_\beta(\lambda + \rho), \theta_7)$

β	e_6	e_7	e_8	$(s_\beta(\lambda + \rho), \theta_7)$
- + + + +	$\frac{a}{2} - 3$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{a-5}{2}$
+ - + + +	$\frac{a-5}{2}$	-1	1	$\frac{a-3}{2}$
+ + - + +	$\frac{a}{2} - 2$	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{a-1}{2}$
+ + + - +	$\frac{a-3}{2}$	-2	2	$\frac{a+1}{2}$
+ + + + -	$\frac{a}{2} - 1$	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{a+3}{2}$
- - - + +	$\frac{a-3}{2}$	-2	2	$\frac{a+1}{2}$
- - + - +	$\frac{a}{2} - 1$	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{a+3}{2}$
- - + + -	$\frac{a-1}{2}$	-3	3	$\frac{5}{2}$
- + - - +	$\frac{a-1}{2}$	-3	3	$\frac{5}{2}$
- + - + -	$\frac{a}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$	$\frac{a+7}{2}$
- + + - -	$\frac{a+1}{2}$	-4	4	$\frac{a+9}{2}$
+ - - - +	$\frac{a}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$	$\frac{a+7}{2}$
+ - - + -	$\frac{a+1}{2}$	-4	4	$\frac{a+9}{2}$
+ - + - -	$\frac{a}{2} + 1$	$-\frac{9}{2}$	$\frac{9}{2}$	$\frac{a+11}{2}$
+ + - - -	$\frac{a+3}{2}$	-5	5	$\frac{a+13}{2}$
- - - - -	$\frac{a}{2} + 2$	$-\frac{11}{2}$	$\frac{11}{2}$	$\frac{a+15}{2}$

If $z \in \{12, 14, 15, 16\}$, then $a \in \{-5, -3, -2, -1\}$ because $a = z - 17$. Thus for all choices of values of $v(i)$ for $2 \leq i \leq 5$, $\frac{1}{2}(-1)^{v(2)} + (-1)^{v(3)} + \frac{3}{2}(-1)^{v(4)} + 2(-1)^{v(5)} - 11 - a \in \mathbb{Z} \setminus \{0\}$. This shows that $s_{e_8-e_7}(\lambda + \rho)$ is Φ_{Γ} -regular integral, and $Y(s_{e_8-e_7}(\lambda + \rho)) \neq 0$ by Proposition 2.2(1). \square

Proposition 3.21. Suppose $\lambda = ae_6 - \frac{1}{2}ae_7 + \frac{1}{2}ae_8$ for some $a \in \mathbb{C}$. Write $\lambda = \lambda_0 + z\zeta$ in the special line. If $z \in \{10, 11\}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible.

Proof. Recall that $\lambda + \rho = e_2 + 2e_3 + 3e_4 + 4e_5 + (a+5)e_6 - \frac{a+17}{2}e_7 + \frac{a+17}{2}e_8$. Assume $z = 10$ first, and then $a = z - 17 = -7$. Recall in the proof of Proposition 3.20 that $(s_{e_8-e_7}(\lambda + \rho), \alpha) = \frac{1}{2}(-1)^{v(2)} + (-1)^{v(3)} + \frac{3}{2}(-1)^{v(4)} + 2(-1)^{v(5)} - 4$ for $\alpha = \alpha_-(v) \in \Phi_{\Gamma}^+$, and if $v(1) = v(2) = 1$ and $v(3) = v(4) = v(5) = 0$, then $(s_{e_8-e_7}(\lambda + \rho), \alpha) = 0$. Thus $s_{e_8-e_7}(\lambda + \rho)$ is Φ_{Γ} -singular. Next consider $s_{e_6 \pm e_i}(\lambda + \rho)$ for $1 \leq i \leq 5$. If $i \neq 3$, then $(s_{e_6 \pm e_i}(\lambda + \rho), e_2 \pm e_i) = 0$. Hence $s_{e_6 \pm e_i}(\lambda + \rho)$ is Φ_{Γ} -regular only if $i = 3$. But $\langle \lambda + \rho, e_6 \pm e_3 \rangle \notin \mathbb{Z}^+$, so $e_6 \pm e_3 \notin S_{\lambda}$. Therefore,

we only need to consider the roots in $\Phi_{\mathfrak{u}}^+$ of the form $\alpha_+(v)$ with $\sum_{i=1}^5 v(i)$ odd. Table 4 lists

the coefficients of e_i for $1 \leq i \leq 5$ of $s_\beta(\lambda + \rho)$ for $\beta \in \{\alpha_+(v) \mid \sum_{i=1}^5 v(i) \text{ odd}\}$. We exclude the case “- - - -” in Table 4 because the root represented by it does not lie in S_{λ} for both $z = 10$ and $z = 11$. Moreover, for $z = 10$, i.e., $a = -7$, the root represented by “+ + - - -” is also excluded for the same reason. Now according to Table 4, one checks immediately that there does not exist $e_j \pm e_i$ for $1 \leq i < j \leq 5$ such that $(s_\beta(\lambda + \rho), e_j \pm e_i) = 0$ only if β is represented by one of the following five sign patterns

$$- + + + +, \quad + - + + +, \quad - - - + +, \quad - - + - +, \quad - - + + - . \quad (3.7.1)$$

Table 4: Coefficients of e_i for $1 \leq i \leq 5$ of $s_\beta(\lambda + \rho)$

β	e_1	e_2	e_3	e_4	e_5
$- + + + +$	$\frac{a}{2} + 8$	$-\frac{a}{2} - 7$	$-\frac{a}{2} - 6$	$-\frac{a}{2} - 5$	$-\frac{a}{2} - 4$
$+ - + + +$	$-\frac{a+15}{2}$	$\frac{a+17}{2}$	$-\frac{a+11}{2}$	$-\frac{a+9}{2}$	$-\frac{a+7}{2}$
$+ + - + +$	$-\frac{a}{2} - 7$	$-\frac{a}{2} - 6$	$\frac{a}{2} + 9$	$-\frac{a}{2} - 4$	$-\frac{a}{2} - 3$
$+ + + - +$	$-\frac{a+13}{2}$	$-\frac{a+11}{2}$	$-\frac{a+9}{2}$	$\frac{a+17}{2}$	$-\frac{a+5}{2}$
$+ + + + -$	$-\frac{a}{2} - 6$	$-\frac{a}{2} - 5$	$-\frac{a}{2} - 4$	$-\frac{a}{2} - 3$	$\frac{a}{2} + 10$
$- - - + +$	$\frac{a+13}{2}$	$\frac{a+15}{2}$	$\frac{a+17}{2}$	$-\frac{a+7}{2}$	$-\frac{a+5}{2}$
$- - + - +$	$\frac{a}{2} + 6$	$\frac{a}{2} + 7$	$-\frac{a}{2} - 4$	$\frac{a}{2} + 9$	$-\frac{a}{2} - 2$
$- - + + -$	$\frac{a+11}{2}$	$\frac{a+13}{2}$	$-\frac{a+7}{2}$	$-\frac{a+5}{2}$	$\frac{a+19}{2}$
$- + - - +$	$\frac{a+11}{2}$	$-\frac{a+9}{2}$	$\frac{a+15}{2}$	$\frac{a+17}{2}$	$-\frac{a+3}{2}$
$- + - + -$	$\frac{a}{2} + 5$	$-\frac{a}{2} - 4$	$\frac{a}{2} + 7$	$-\frac{a}{2} - 2$	$\frac{a}{2} + 9$
$- + + - -$	$\frac{a+9}{2}$	$-\frac{a+7}{2}$	$-\frac{a+5}{2}$	$\frac{a+15}{2}$	$\frac{a+17}{2}$
$+ - - - +$	$-\frac{a}{2} - 5$	$\frac{a}{2} + 6$	$\frac{a}{2} + 7$	$\frac{a}{2} + 8$	$-\frac{a}{2} - 1$
$+ - - + -$	$-\frac{a+9}{2}$	$\frac{a+11}{2}$	$\frac{a+13}{2}$	$-\frac{a+3}{2}$	$\frac{a+17}{2}$
$+ - + - -$	$-\frac{a}{2} - 4$	$\frac{a}{2} + 5$	$-\frac{a}{2} - 2$	$\frac{a}{2} + 7$	$\frac{a}{2} + 8$
$+ + - - -$	$-\frac{a+7}{2}$	$-\frac{a+5}{2}$	$\frac{a+11}{2}$	$\frac{a+13}{2}$	$\frac{a+15}{2}$

Therefore, we only need to consider these five roots, which are the only possible roots β such that $s_\beta(\lambda + \rho)$ are Φ_{Γ} -regular in S_λ .

Let us consider the root represented by “ $- + + + +$ ”, i.e.,

$$\beta_0 = \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - e_7 + e_8).$$

First, $\langle \lambda + \rho, \beta_0 \rangle = 9 \in \mathbb{Z}_{>0}$ shows that $\beta_0 \in S_\lambda$. Second, according to Table 4, we know that

$$s_{\beta_0}(\lambda + \rho) = \frac{1}{2}(9e_1 - 7e_2 - 5e_3 - 3e_4 - e_5 - 13e_6 - e_7 + e_8).$$

It is obvious that $(s_{\beta_0}(\lambda + \rho), e_j \pm e_i) \in \mathbb{Z} \setminus \{0\}$ for $1 \leq i < j \leq 5$. For a root of the form $\alpha_-(v)$ in Φ_{Γ}^+ ,

$$(\alpha_-(v), s_{\beta_0}(\lambda + \rho)) = \frac{1}{4}(9(-1)^{v(1)} - 7(-1)^{v(2)} - 5(-1)^{v(3)} - 3(-1)^{v(4)} - (-1)^{v(5)} + 15)$$

which equals 0 if and only if $v(1)$ and $v(3)$ are even, while $v(2)$, $v(4)$, and $v(5)$ are odd. But this is not a root in Φ_{Γ} . This shows that $s_{\beta_0}(\lambda + \rho)$ is Φ_{Γ} -regular integral. Hence $Y(s_{\beta_0}(\lambda + \rho)) \neq 0$ by Proposition 2.2(1).

According to Table 3, we know that if $\beta_1 \neq \beta_2$ where β_1 and β_2 are chosen from the five roots in (3.7.1), then $(\beta_1, \theta_7) \neq (\beta_2, \theta_7)$. Hence by Lemma 3.19, there does not exist $\omega \in W_{\Gamma}$ such that $s_{\beta_0}(\lambda + \rho) = \omega s_{\beta}(\lambda + \rho)$ for any other root β chosen from the five roots in (3.7.1). By Proposition 2.4 and Theorem 2.3, the conclusion holds in the case of $z = 10$.

The proof for $z = 11$ is parallel. One shows that the only possible roots β such that $s_{\beta}(\lambda + \rho)$ are Φ_{Γ} -regular in S_λ are just those represented by “ $- + + + +$ ”, “ $+ - + + +$ ”, “ $+ + - + +$ ”, “ $+ + + - +$ ”, and “ $+ + + + -$ ”, and the root represented by “ $+ - + + +$ ” can be chosen as β_0 as in the case of $z = 10$. \square

Theorem 3.22. *Suppose $\lambda = ae_6 - \frac{1}{2}ae_7 + \frac{1}{2}ae_8$ for some $a \in \mathbb{C}$. Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $a \in -8 + \mathbb{Z}_{\geq 0}$.*

Proof. Write $\lambda = \lambda_0 + z\zeta$ in the special line, and then Lemma 3.18'(1) and Theorem 2.6(1) show that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible only if $z \in 9 + \mathbb{Z}_{\geq 0}$. On the other hand, Lemma 3.18'(2), Proposition 3.20, Proposition 3.21, and Theorem 2.6(2) show that if $z \in 9 + \mathbb{Z}_{\geq 0}$, then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible. It follows that $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is reducible if and only if $z \in 9 + \mathbb{Z}_{\geq 0}$, which is equivalent to $a \in -8 + \mathbb{Z}_{\geq 0}$. \square

Acknowledgements

I felt interested in the reducibility of generalized Verma modules when discussing with Doctor Zhanqiang BAI on another research problem which inspired me with the ideas. In the process of the study, Doctor Toshihisa KUBO sent me his Ph.D. thesis which offered me some useful techniques. After I finished the paper, some reviewers gave me very concrete and thoughtful advice on revising the paper. I would like to express my thankfulness to all of them.

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